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EXTENSION OF CHERNIKOVA'S ALGORITHM FOR SOLVING GENERAL MIXED LINEAR PROGRAMMING PROBLEMS

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Extension of Chernikova's Algorithm for Solving General Mixed Linear Programming Problems

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Abstract

The purpose of this paper is to develop an algorithm for determining a general solution of a mixed system of linear inequalities and equations without assuming any additional condition. The method is an extension of Chernikova's algorithm (non-negative domain), and is essentially based on the concept of duality on polyhedral cones. The same algorithm allows us to solve the dual problem: determination of a set of inequalities and equations which characterizes a given polyhedron. The proposed algorithm is especially adequate in applications with relatively small problem matrices, and where exact general solutions are required: CAD systems, computational geometry, piecewise linear modeling, etc. Some simple examples are given to illustrate some definitions and properties, and the main steps of the algorithm.

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EXTENSION DE L'ALGORITHME DE CHERNIKOVA POUR POUR LA RESOLUTION DE PROBLEMES DE PROGRAMMATION LINEAIRE MIXTES

Résumé

L'objet de ce rapport est de développer un algorithme pour la détermination de la solution générale d'un système mixte d'inéquations et d'équations linéaires sans supposer de conditions supplémentaires. La méthode est une extension de l'algorithme de Chernikova (valable pour domaine non négatif). Elle est essentiellement basée sur le concept de dualité de cônes polyédriques. L'algorithme permet aussi de résoudre le problème dual : déterminer l'ensemble d'inéquations et d'équations qui caractérisent un polyèdre donné. L'algorithme proposé est particulièrement bien adapté aux applications ayant une matrice de contraintes de petite taille et pour lesquels une solution exacte est nécessaire, systèmes de conception assistée, géométrie computationnelle, modélisation par fonction linéaire par morceaux, etc. Des exemples simples sont donnés pour illustrer les définitions et propriétés et les principales étapes de l'algorithme.

2 Review of the problem

Algorithms for determining a dual representation of a convex polyhedron (finding a fundamental set of all solutions) can be divided into two main classes: *pivoting* and *nonpivoting* methods.

The pivoting methods are based on the application of the simplex method. Most of them assume that the solution polyhedron is bounded (polytope), nondegenerated and placed in the nonnegative orthant, although they can be modified by standard techniques to overcome these difficulties [LEMAIRE 88]. However these schemes complicate the algorithm and make it less efficient.

The nonpivoting methods can be viewed as derived from the Double Description Method [MOTZKIN 53] and are dual of the Fourier-Motzkin elimination method [CHVATAL 83] [SCHRIJVER 87] (where the variables are successively eliminated). This last method can be viewed as a sequence of projections of the corresponding polyhedron; the complexity of the algorithm is not polynomial [SCHRIJVER 87].

Most of the pivoting algorithms assume that the polyhedron is contained in the nonnegative orthand ($x \geq 0$), in this case the polyhedron is pointed and its lineality space (the largest affine subspace it contains) must have dimension 0, and the solution polyhedron is defined by vertices and extremal rays (it does not contain lines [CHVATAL 83]).

Many of the nonpivoting methods are originally stated in terms of finding all extreme rays of a pointed convex polyhedral cone defined by a system of homogeneous linear inequalities, with nonnegativity constraints:

$$Ax \geq 0 \qquad (1)$$

$$x \geq 0$$

where A is an $m \times n$-matrix and x a column n-vector.

However, the system of nonhomogeneous linear inequalities:

$$Ax \geq b \tag{2}$$
$$x \geq 0$$

where b is a column n-vector, can be transformed into an equivalent homogeneous form:

$$[A| - b]\ xh \geq 0 \tag{3}$$
$$xh \geq 0$$

where $[A| - b]$ is a $m \times (n+1)$-matrix and xh a column $(n+1)$-vector corresponding to the homogeneous coordinates of the vector x.

The set of extremal ray $(n+1)$-vectors, solution to the previous system, may be separated into two disjoint subsets, which correspond to

1) Extremal rays: having $xh_{n+1} = 0$
2) Vertices: having $xh_{n+1} > 0$

of the original nonhomogeneous system (2).

One advantage of using homogeneous coordinates is that the general solution of a linear constraint system may be found for bounded or unbounded polyhedra in a unified manner. A second advantage is that the duality between elements of the polyhedral cone can be defined in complete form.

The transformation can be also made conversely, in order to more clearly see some properties of the polyhedral cones in an n-dimensional space, as an equivalent one of the polyhedra in an $(n-1)$-dimensional space.

The Chernikova algorithm, which is the basis of presented algorithm, falls into this class of nonpivoting methods, and allows us to find a general expression for the non-negative solutions of a system of linear constraints (inequalities or equations).

In Chernikova's approach, the pointed convex cone of all negative vectors $\mathcal{Q}_+^n$, is taken as initial solution cone. This cone $\mathcal{Q}_+^n$ is self-dual (the concept of duality and other cone-theoretic concepts are precisely specified later).

When the constraints of nonnegativity ($x \geq 0$) are not necessarily present (hypothesis accepted in this paper), the situation is rather more complex. In this case the nonpointed convex cone of all vectors $\mathcal{Q}^n$ is taken as the initial solution cone. This cone $\mathcal{Q}^n$ has as dual the convex cone $\{0\}$.

Basically, Chernikova's algorithm obtains the final solution by means of a sequence of transformations of this original cone, adding linear equalities or equations in a stepwise manner. Each cone differs from its predecessor in having one additional constraint. In each step, the algorithm finds the new cone, by computing the new extremal rays and by excluding some of the old ones. The total amount of calculation depends on the order in which the inequalities (or equations) are processed [CHERNIKOVA 65] [RUBIN 75].

It is convenient to remark that the central problem of the nonpivoting methods is: to assure the elimination or to avoid the generation of redundant (nonextreme, in the case of a pointed cone) rays at each step of the algorithm, rather than eliminate them at the end of the process.

The nonpivoting methods, where the redundancy of rays is either unconsidered [SOLODONIKOV 77][MOO 87], or partially considered [UZAWA 58][UZDAVA 62][SILVA 85] generate a nonminimal solution, they are generally simpler, but computationally less efficient, and require more storage.

Moreover, a simple condition (the key of Chernikova's algorithm) can be used to characterize the extremal rays of a pointed polyhedral cone. A ray is extremal iff no other ray of this cone verifies the same set of boundary hyperplanes that this ray verifies (unique solution). In equivalent form: two extremal rays are adjacent iff no other ray is contained in the intersection of boundary planes (minimal face) of the cone which they verify (a more general case will be described later).

Surveys of the literature on this topic are given in the references [MATTHEISS 80] [SHRIJVER 86] [DYER 76]. A description of Chernikova's algorithm appears in [CHERNIKOVA 64,65,68] [MATTHEISS 80] [RUBIN 75].

In the present approach, where the nonnegativity constraints are not considered, the fact that the initial cone is not pointed imposes a more general consideration of the theory of polyhedral cones.

In the next sections some properties of general polyhedral cones are discussed which are instrumental for the subsequent development of the considered algorithm. Also, the standard concept of homogeneous coordinates is revised, as well as the new concept of bidirectional coordinates.

3 Homogeneous Coordinates

In this section we briefly review the homogeneous representations of points and half-spaces.

Homogeneous coordinates of a point or vector $(x_1, \ldots, x_n) \in \mathcal{Q}^n$ are the $(n+1)$-tuples of the equivalent class

$$Homogeneous\ Coordinates\,((x_1, \ldots, x_n)) = \{(px_1, \ldots, px_n, p) : p \neq 0\}$$

the particular element of this class:$(x_1, \ldots, x_n, 1)$, is called homogeneous representation for $(x_1, \ldots, x_n)$.

We call integer homogeneous representation of this class (in the case of a rational point) the tuple

$$(dx_1, \ldots, dx_n, d) \in \mathcal{Z}^{n+1}$$

where d is the least common multiple of the denominators of the coordinates $(x_1, \ldots, x_n)$. The proposition above permits us to obtain a simple integer representation of a rational point, which is convenient to represent and compute exact rational points on a computer in homogeneous form.

The physical coordinates from a homogeneous representation is obtained by the mapping ($x_{n+1} \neq 0$)

$$(x_1, \ldots, x_n, x_{n+1},) \longrightarrow \frac{1}{x_{n+1}}(x_1, \ldots, x_n)$$

A point with homogeneous representation of the form

$$(x_1, \ldots, x_n, 0)$$

where: $x_i \neq 0$ for some i ($1 \leq i \leq n$), corresponds to a point at the infinity. These points can also be considered direction vectors (rays). One advantage of the homogeneous representation is that rays and finite points can be used in a unified manner.

In dual form a half-space

$$a_1x_1 + \cdots + anx_n + b \geq 0$$

is specified in homogeneous coordinates by the equation

$$a_1x_1 + \cdots + anx_n + bx_{n+1} \geq 0 \qquad (x_{n+1} \geq 0)$$

If $a_1 = \cdots = a_n = 0$ and $b \neq 0$ then the corresponding hyperplane is at infinity. In this extended homogeneous space always two hyperplanes intersect at the origin.

The use of homogeneous coordinates allows us to transform a convex polyhedron into a polyhedral cone, so we will restrict our discussion to this kind of convex sets.

4 Bidirectional coordinates

In this homogeneous space to represent rays and half-spaces, the corresponding oriented vectors are used. It is also convenient to have a simple model of representation of the non-directed elements: lines (bidirectional rays) and hyperplanes (equations), which allows us to consider them as unique elements, and avoids the unnecessary duplication of information.

In the development of the algorithm, we use the new concept of a bidirectional ray, i. e. an entity like a ray, but allowed to contain two orientations simultaneously.

The bidirectional representation of a ray is defined by the following mapping:

$$\begin{array}{lcl} \textit{Bid. Coordinates}\,(\quad x \quad) = & (1, x) & \text{(unidirectional ray vector)} \\ \textit{Bid. Coordinates}\,(\; x, -x \;) = & (0, x) & \text{(bidirectional ray vector)} \end{array}$$

where $x = (x_1, \ldots, x_n)$ is a directional vector of an homogeneous space of dimension n.

This representation allows us to study directed and undirected elements, i. e. half-spaces (inequalities) and hyperplanes (equations) in a unified manner and also permits a substantial simplification of the algorithm proposed. Moreover, by working with bidirectional rays, the duality between hyperplans (equations) and bidirectional rays is conserved.

Example: The inequality

$$(3\;2)\;\;(x_1\;x_2)^T \geq 0$$

where v^T denotes the transpose of vector v, can be represented in bidirectional coordinates by the inequality

$$((1)\;3\;2)\;\;((\mu)\;x_1\;x_2)^T \geq 0$$

where μ is the coordinate of bidirectionality.

Analogously the equation

$$(3\ 2)\ \ (x_1\ x_2)^T = 0$$

can be represented in bidirectional coordinates by the inequality

$$((0)\ 3\ 2)\ \ ((\mu)\ x_1\ x_2)^T \geq 0$$

□

The next sections deal with some basic definitions and properties of polyhedral cones, which are the bases of the considered algorithm.

5 Convex Polyhedral Cones

From the Finite Basis Theorem of Minkowski (1896), a convex polyhedral cone can be defined in two ways:

- In *implicit* form: as the common solution of a set of linear inequalities:

$$C = \{x : Ax \geq 0\} \tag{4}$$

  where x is a column n-vector and A an $(m \times n)$-matrix of constraint rows (inequalities);

- In *parametric* form: as the nonnegative linear combination of a set of rays

$$\begin{aligned} C &= \{y_1 r^1 + \cdots + y_{m1} r^{m1}\ :\ y_1, \ldots, y_{m1} \geq 0\} \\ &= \{x\ :\ x = R\,y\ ,\ y \geq 0\} \end{aligned} \tag{5}$$

where y is a column $m1$-vector and R is an $(n \times m1)$-matrix of column rays which contains a fundamental set of rays of the cone.

A cone C is pointed iff $C \cap (-C) = \{0\}$ ($\{0\}$ is the only subspace contained in the cone), i. e. iff it does not contain lines (bidirectional rays) [CHVATAL 83]. In the case of a pointed cone, the fundamental set (unique) which generates all the rays of the cone, is the set of its extremal rays.

The dual cone C^* of a polyhedral cone C, also named polar cone or non-negative dual cone, is a polyhedral cone defined by

$$C^* = \{x^* \; : \; x^* \, x^T \geq 0 \, , \; x \in C\}$$

The theorem of duality for a convex polyhedral cone shows:

$$(C^*)^* = C$$

Therefore, the polyhedral cone generated by the row vectors of the constraint matrix A of the system (4), is the dual of the cone generated column vectors of the matrix R of the system (5).

The concept of duality (symmetric relation) is a powerful tool for the description, analysis and construction of algorithms. By applying it to theorems, proofs, and algorithms we can obtain twice as many results with practically the same effort.

Particular common dual polyhedral cones are given in Table 1.

Two subspace polyhedral cones $W1$ and $W2$ of a vector space V are dual (complementary) iff $W1 \cap W2 = \{0\}$ and $W1 + W2 = V$.

In the sequel we shall make frequent use of dual transformations (dualization) to deduce some of the considered properties.

Cone	Dual cone
Half-space	Ray (Unidirectional ray)
Hyperplane	Line (Bidirectional ray)
Pointed cone	Full-dimensional cone
Q^n_+	Q^n_+
Q^n	$\{0\}$
Subspace	Complementary subspace

Table 1: Duality between some polyhedral cones.

6 Decomposition of Polyhedral cones

If a polyhedral cone C is implicitly defined as the solution of a System of linear inequalities (4), it can be decomposed into two polyhedral cones: C^0 and C^1 [SCHRIJVER 86] such that

$$C = C^0 + C^1 \tag{6}$$

where

$$\begin{array}{l} C^0 = \{x \,:\, A^0\, x \geq 0\} \;\; = \{x \,:\, A^0\, x = 0\} \\ C^1 = \{x \,;\, A^1\, x \geq 0\} \end{array} \tag{7}$$

$A^0\ x \geq 0$ is the subsystem of implicit equations in $A\ x \geq 0$ (which defines the contained subspace in C), and $A^1\ x \geq 0$ is the subsystem of all other inequalities in $A\ x \geq 0$ (which defines a maximal pointed cone contained in C).

In dual form, in accordance with the parametric definition (5) of a cone C, this decomposition can also be expressed as

$$\begin{array}{l} C^0 = \{x \,:\, x = R^0\, y \;,\; y \geq 0\} \;\; = \{x \,:\, x = R^0\, y\} \\ C^1 = \{x \,:\, x = R^1\, y \;,\; y \geq 0\} \end{array} \tag{8}$$

where R^0 is the submatrix of implicit bidirectional column rays (which defines the contained subspace in C), and R^1 is the submatrix of all other rays in R (which defines a maximal pointed cone contained in C).

It is convenient to note that the submatrix of implicit bidirectional column vectors R^0, and the submatrix of implicit equations A^0, can be greatly simplified (half reduction) if bidirectional coordinates are used. In addition, the existence of equalities or bidirectional rays can be expressed explicitly by the value of the corresponding coordinate of bidirectionality (main advantage).

Example: The polyhedral cone $\{0\}$ in a space of dimension 3 (dual of the cone Q^2) can be defined, using bidirectional coordinates, as the solution of the system of linear inequalities (theorem of Minkowski):

$$\begin{bmatrix} (0) & 1 & 0 \\ (0) & 0 & 1 \end{bmatrix} \begin{bmatrix} (\mu) \\ x_1 \\ x_2 \end{bmatrix} \geq 0$$

Analogously, the cone Q^2 can be defined in parametric form as:

$$\begin{bmatrix} (\mu) \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (0) & (0) \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq 0$$

□

A half-space or linear inequality of a system is called redundant in the system iff it is implied by the other inequalities in the system (it is a nonnegative linear combination of the rest of the inequalities).

Dualizing, a ray of a set is called redundant in the set, iff it is generated by the other rays in the set (it is a nonnegative linear combination of the rest of the rays).

A system of inequalities (or set of ray vectors) is irredundant iff its has no redundant elements. A subsystem of inequalities (or a subset of ray vectors) of a polyhedral cone is called fundamental, iff it defines the corresponding cone and is irredundant. The cardinality of any fundamental subset of a cone is not necessarily the same. If bidirectional coordinates are used the cardinality of any fundamental system of inequalities (or set of ray vectors) will be the same (minimal normalization).

Example: The cone Q^2 can be alternatively defined as a nonnegative linear combination of row ray vectors belonging to the two following fundamental sets Ra and Rb

$$Ra = \{(x_1, x_2) : \quad (1,0)\ ,\ (0,1)\ ,\ (-1,-1) \quad \}$$

$$Rb = \{(x_1, x_2) : \quad (\pm 1,0)\ ,\ (0,\pm 1) \qquad \}$$

□

7 Face-lattice of a polyhedral cone

A subset (subcone) F of a polyhedral cone C is called a face of C, iff $F = C$ or there exists an inequality of $a^T\ x \geq 0$ which is valid for C, i. e. $C \subseteq \{x\ :\ a^T\ x \geq 0\}$, such that

$$F = \{x \in C\ :\ a^T\ x = 0\}$$

It is possible to describe in detail the structure of a polyhedral cone by a partial ordered set of faces of different dimensions. Every k-dimensional face is also a cone union of a finite number of faces of dimension $k-1$, with the condition that each pair of them is not placed in the same plane of dimension $k-1$ [BOLOTIANSKI 76].

The intersection of two faces is empty or a face again. Hence, the faces form a lattice under inclusion (partial order) which is called the face-lattice of C. The Hasse diagram of the face-lattice of a cone displays the structure of the cone and facilitates the study of the properties of any polyhedral cone in schematical form. The lattice of a polyhedral cone C contains a unique face of the subspace L (lineality space) of dimension l. The faces of dimension $l+1$ are called minimal proper faces [SCHRIJJVER 86]. If C is pointed ($l = 0$) the minimal proper faces are called extremal rays from C. A facet of C is a maximal face distinct of C.

Two minimal proper faces of C are adjacent if they are contained in only one face of dimension $l + 1$.

Example: Table 2 indicates the Hasse diagrams of some common polyhedral cones, read horizontally. The dimension of each face is denoted between parenthesis, and the letters L and C denotes: C=Cone, L=Lineality space.

Polyhedral Cone	**Hasse diagram**
Subspace	$L = C$ • (Isolated point of the same dimension)
Halfspace	L (2) • $\longrightarrow$ • (3) C
Ray	L (0) • $\longrightarrow$ • (1) C
Dihedral angle	• (2) A L (1) • < > • (3) C (A, B: 2−faces) • (2) B

Table 2: Horizontal Hasse diagrams of some polyhedral cones in Q^3.

□

In the next sections of this paper, the main stages of the proposed algorithm are developed.

8 Initial matrix

The problem we are precisely concerned is to find a general solution of the linear homogeneous system:

$$C = \{x \ : \ B\,x = 0 \ , \ D\,x \geq 0\}$$

where B and D are matrices of dimensions $m1 \times n$ and $m2 \times n$ respectively, and x a column n-vector.

If the bidirectional coordinates are used this system can be put in form of

$$C = \{x \ : \ A\,xb \geq 0\,\}$$

where

$$A \ = \ \left[\begin{array}{c|c} (0) & B \\ \hline (1) & D \end{array}\right]$$

is a $m \times (n+1)$–matrix ($m = m1 + m2$), and $xb = [(\mu) \ x^T]^T$ a column $(n+1)$–vector (the first component is the bidirectionality coordinate).

The following $((n+1) \times (n+1))$–matrix R corresponds to the initial ray matrix:

$$R \ = \ I$$

where I is the identity $((n+1) \times (n+1))$–matrix, which represents the set of bidirectional row ray $(n+1)$–vectors which generate the initial cone Q^n.

As in Chernikova's algorithm, the following composed $((n+1) \times (n+1) + m)$–matrix T is used:

$$T \;=\; [R \mid A^T] \;=\; [t_{i,j}]$$

The coefficients of the columns of the matrix A^T represent the projection of the corresponding oriented row ray vector (the positive part if it is a bidirectional ray (or a line)) on the corresponding oriented column hyperplane (the positive part if it is bidirectional (or an equation)).

$$t_{ij} \; = < r^i \; , \; a^j >$$

i. e., the inner product of the ray $n-$vector r^i by the $n-$vector a^j which is orthogonal to the corresponding hyperplane.

Example: The initial composed matrix (tableau) of the linear mixed systems of constraints:

$$\begin{array}{lrcr} a) & z & = & 1 \\ b) & x & \geq & 0 \\ c) & -x & \geq & -1 \\ d) & y & \geq & -1 \end{array}$$

is

$$T \;=\; [R \,||\, A^T] \;=\; \left[\begin{array}{c|c|cccc|c||ccccc} & \mu & x & y & z & \lambda & a) & b) & c) & d) & e) \\ \hline \mu & (1) & (0) & (0) & (0) & (0) & (0) & (1) & (1) & (1) & (1) \\ \hline x & (0) & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ y & (0) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ z & (0) & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \lambda & (0) & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 1 & 1 \end{array}\right]$$

where λ is the homogeneous coordinate, and μ is the bidirectional coordinate.

The column $e)$ corresponds to the inequality $(\lambda \geq 0)$. The bidirectionality coordinate of rays μ_R and constraints μ_H is indicated between parenthesis.

μ_R: ((0) = line, (1) = ray). μ_H: ((0) = equation, (1) = inequality).

□

9 Computation of New Ray Vectors

The algorithm computes a succession of transformations on the initial matrix T to obtain a fundamental set of rays which generates the solution cone.

At each step, a hyperplane or constraint is selected. A heuristic for this selection is given posteriorly.

If we have a cone C, in wich one fundamental set of rays are already known, the new cone C' is obtained from C by adding another constraint H, that is, C' is the intersection of C with the closed half-space (inequality) H^1 ($\mu_H = 1$), or with the hyperplane (equation) H^0 ($\mu_H = 0$).

The rays of C' are some of the rays of C:

- In the case of an inequality, those in H^1 (whose projections are positive or null).

- In the case of an equation, those on H^0 (whose projections are null).

and some new rays obtained as positive combination of a pair of rays.

Each ray of a pair belongs to one different side of the considered hyperplane: the open half-space H^+ (positive projection rays) and the opposite open half-space H^- (negative projection rays). The obtained new rays all must lie on the the hyperplane H^0.

Let H^0: $a^T\ x = 0$, be the equation of the considered hyperplan; and let r^p be a ray with positive projection, and let r^n be a ray with negative projection, which belong to the previous cone C. This rays belong to the open half-spaces: H^+ and H^- respectively. The new ray holds

$$r = \alpha_1 \, r^p + \alpha_2 \, r^n$$

where α_1 and α_2 are positive numbers. This new ray also verifies the hyperplane H^0:

$$a^T \, [\, \alpha_1 \, r^p + \alpha_2 \, r^n \,] = 0$$

A solution of that condition is:

$$\begin{aligned} \alpha_1 &= -k \, a^T \, r^n \\ \alpha_2 &= k \, a^T \, r^p \end{aligned}$$

where k is any arbitrary positive constant. If the considered system is integer, in order to simplify the resultant ray, k can be made equal to:

$$k = \frac{1}{g.c.d.(a^T \, r^n \, , \, a^T \, r^p)}$$

Analogously, the obtained new ray vector can also be simplified, dividing it by the greatest common divisor (g.c.d) of its components.

Due to the distributive property of the inner product, the new rows of the matrix T, are successively obtained by positive linear combinations of the corresponding rows.

If the bidirectionality of the rays is considered ($\mu_R = (0)$ or (1)), the following additional rules must be taken into account, depending on the value of the projection of the ray on the considered hyperplane (positive, negative or null):

- A bidirectional ray with ***null*** projection is transformed into the same bidirectional ray.

- A bidirectional ray with ***positive*** projection is transformed into the same unidirectional ray,

- A bidirectional ray with ***negative*** projection is transformed into the opposite unidirectional ray.

- A linear combination of a *unidirectional* ray with a *bidirectional* ray is an unidirectional ray, obtained by a linear combination of the unidirectional ray with the part of the bidirectional ray which has the opposite projection.

- A linear combination of ***two bidirectional*** rays is another bidirectional ray, obtained by considering the positive part of one and the negative part of the other (indistinctly).

Example: For the system of linear inequalities:

$$a)\ x \geq y \quad b)\ x \geq 0 \quad c)\ \lambda \geq 0$$

the initial matrix T will be:

$$T = [R \,||\, A^T] = \left[\begin{array}{c|c|cc|c||ccc} & \mu & x & y & \lambda & a) & b) & c) \\ \hline \mu & (1) & (0) & (0) & (0) & (1) & (1) & (1) \\ \hline x & (0) & 1 & 0 & 0 & 1 & 1 & 0 \\ y & (0) & 0 & 1 & 0 & -1 & 0 & 0 \\ \lambda & (0) & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

If the inequality $a)$ is selected, the new matrix T' of the conserved rays will be:

$$T' = \left[\begin{array}{c|c|cc|c||ccc}
 & \mu & x & y & \lambda & a) & b) & c) \\
\hline
\mu & (1) & (0) & (0) & (0) & (1) & (1) & (1) \\
\hline
x & (1) & 1 & 0 & 0 & 1 & 1 & 0 \\
-y & (1) & 0 & -1 & 0 & 1 & 0 & 0 \\
\lambda & (0) & 0 & 0 & 1 & 0 & 0 & 1
\end{array}\right]$$

Adding the new rays of the cone, which are the linear combination of positive and negative projection rays, we obtain the new matrix T'' formed by the rows of the conserved and new rays:

$$T'' = \left[\begin{array}{c|c|cc|c||ccc}
 & \mu & x & y & \lambda & a) & b) & c) \\
\hline
\mu & (1) & (0) & (0) & (0) & (1) & (1) & (1) \\
\hline
x & (1) & 1 & 0 & 0 & 1 & 1 & 0 \\
-y & (1) & 0 & -1 & 0 & 1 & 0 & 0 \\
\lambda & (0) & 0 & 0 & 1 & 0 & 0 & 1 \\
x+y & (0) & 1 & 1 & 0 & 0 & 1 & 0
\end{array}\right]$$

□

10 Fundamental Set of Ray Vectors

The new set of ray vectors obtained by the previous procedure is in general a nonfundamental set of rays (redundant set).

Rather than eliminate the redundant rays at the end of the process, it is convenient to avoid generating them at each step of the algorithm, by considering an additional criterion.

The following two theorems give us simple conditions to characterize a fundamental set of rays of a general polyhedral cone.

Theorem 1 . *The set of unidirectional rays R^1 are irredundant in the set of rays R ($R = R^1 \cup R^0$) of a cone C iff for all distinct unidirectional rays r^α and r^β from R^1 there is an inequality $a^T\ x \geq 0$ which verifies the cone C such that $a^T\ r_\alpha = 0$, $a^T\ r_\beta > 0$.*

Proof: This theorem is the dual of the corresponding to a linear system of irredundant inequalities. A proof of the dual theorem appears in [SCHRIJVER 86].

□

A particular situation of interest is when there is only one unidirectional ray vector in R^1 which will be irredundant.

In the case of a pointed cone, there is no bidirectional rays ($R = R^1$), and the following corollary holds.

Corollary 1 . *If C is a pointed cone, then the set of ray vectors R which defines C is fundamental or irredundant (are extremal) iff for all distinct unidirectional rays r^α and r^β from R^1 there is an inequality $a^T\ x \geq 0$ which verifies the cone C such that $a^T\ r_\alpha = 0$, $a^T\ r_\beta > 0$.*

This last condition is coincident with the criterion applied in Chernikova's algorithms in the detection of extremal rays.

In the general case (nonpointed cone) an additional criterion is necessary in order to obtain a fundamental set of bidirectional ray vectors (independent set of vectors). The next theorem give us a simple procedure to obtain a fundamental set of bidirectional vectors.

Theorem 2 *Given a fundamental set of (independent) bidirectional vectors R^0 of a cone C, a new set of fundamental bidirectional vectors R'^0 of the new cone C', obtained by adding a new inequality $a^T\ x \geq 0$, can be determined by:*

1) *Taking all null-projection bidirectional rays of R^0.*

2) *Computing the corresponding linear combinations of one (positive or negative)- projection bidirectional ray of with the rest of (positive or negative)-projection bidirectional rays (a projection transformation).*

Proof: It follows directly from the properties of vector spaces. If the corresponding Hyperplane intersects the lineality space of the original cone of dimension n, then the new lineality space has a dimension equals to $n-1$. Moreover the new set of $n-1$ rays computed constitutes a basis of the new subspace because they are independent by construction.

Example: A fundamental set of the previous example is obtained by excluding one of the two unidirectional row ray vectors:

$$((1)\,1\,0\,0) \quad ((1)\,0\,-1\,0)$$

because they do not satisfy the theorem 1 (redundant set).

□

11 Enhancement of the Algorithm

In order to facilitate the generation of a new fundamental set of unidirectional ray vectors, it is convenient to order the rays of the cone according to the number of considered inequalities which they verify. This number is called *index* of that ray.

By this way, if the rays of the cone are ordered in increasing order of index, the verification of irredundancy of a given unidirectional ray r can be made by comparing r with the rest of old unidirectional rays which has a index ray greater or equal to the the index ray of r.

12 Order of selection of constraints

The order in which the column constraints of the matrix T are processed can vary greatly the work needed to obtain the solution of any given problem, especially when there is a relatively large number of redundant constraints.

The proposed heuristic of the selection order of constraints is:

- First: process the equations.

- Second: process the inequalities.

For the suborder inside these fields:

1) Compute the equations in any order. The new redundant equations (which have null-projections for each ray of the considered cone C) are not considered.

   In this case, the rays of the successive cones are all bidirectional, since these cones are subspaces. If the cone C has $n1$ irredundant (independent) bidirectional rays, the new cone C' obtained ,adding an additional irredundant equation, will have $n1 - 1$ irredundant bidirectional rays.

2) Compute the unconsidered inequality which restricts more the considered cone C, i. e. the inequality $a_j^T\ x \geq 0$, which verifies that one of the projections of the rays of C on it, has the minimun algebraic value:

$$pmin_j \ = \ min(\frac{a_j^T\ r_i}{|a_j|}) \ = \ min(\frac{t_{ij}}{|a_j|})$$

   where

a_j	$=$	Orthogonal n-vector to the corresponding half-space.
r_i	$=$	Ray n-vector of the cone C.
$\|a_j\|$	$=$	Modulus of the vector a_j.
t_{ij}	$=$	Coefficients of the matrix T.

This algebraic projection $pmin_j$ gives us a measure of the size of the region of the cone C constrained by the half-space j.

An unconsidered inequality of the system will be redundant, if all coefficients of the corresponding column are: null for the bidirectional rays of the cone C, and positive or null for unidirectional rays of the cone C. These irredundant inequalities can be eliminated.

The proposed heuristic of selection of inequalities gives a better order of selection than the myopic heuristic [RUBIN 75], especially when there is a relative big number of redundant inequalities.

13 Description of the algorithm

An sketch of the proposed algorithm to solve a homogeneous system of linear constraints in an $n-$space is:

1. Generation of the initial matrix T:

   T = concatenation (identity ray matrix , transpose constraint matrix)

   (*These two matrix are expressed in bidirectional and homogeneous coordinates*)

2. WHILE Number of unconsidered irredundant constraints is not zero

   DO

   (a) Constraint selection:

IF there are irredundant unconsidered equations
THEN select one.

ELSE IF there are irredundant unconsidered inequalities
THEN select the most restrictive one.

(b) Generate a fundamental set of rays using theorem 1 and 2.

(c) Construction of a new matrix T.

3. Select the first $n+1$ columns of the the matrix T, whose rows are the bidirectional coordinates of a fundamental set of ray vectors of the solution.

14 Conclusion

In this paper a generalization of the Chernikova's Algorithm method to an unrestricted domain, has been presented. This method is mainly based on the concept of duality on polyhedral cones. It allows us obtaining a fundamental set of solutions for any mixed linear system of inequalities and equations.

The consideration of the concept of bidirectional ray, as dual to the concept of implicit equation, permits us a substantial simplification in the process of generation of new irredundant ray vectors.

A heuristic for the order of inequally selection is shown, in order to eliminate the redundant constraints at the first steps of the algorithm.

15 Acknowledgements

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17 Appendix

This section shows two examples (primal & dual cones) of the described method. The examples below are actual outputs from a Pascal program implementing the algorithm.

```
Example of execution of the program: <mat67.p>:
===============================================

Constraint matrix (bidirectional) Q:
 - First component: 0 <=> bidirectional    <=> +/-
                    1 <=> unidirectional. <=>

Matrix Q:
   (1)      0   0   0   1
   (1)     -2   2  -1   0
   (1)      4   2  -1   0
   (1)      2   6  -3 -20
   (1)     -6  10  -5 -24

Initial tableau:

Tableau I:
-7 |   0 |   1   1   1 |   1 ||   1   5   7  31  45 <= Norm
-6 |   0 |  10  10  10 |  10 ||  10  30  46 212 271 <= Modulus (approx.)
-5 |   0 |   0   0   0 |   0 ||   0   0   0   0   0 <= Redundant
-4 |   0 |   0   0   0 |   0 ||   0   0   0   0   0 <= Considered
-3 |   0 |   6   6   6 |   6 ||   6   2   2   0   0 <= NO of (0)
-2 |   0 |   1   1   1 |   1 ||   1   3   3   4   4 <= NO of (+)
-1 |   0 |   1   1   1 |   1 ||   1   3   3   4   4 <= NO of (-)
 0 |   0 |   0   0   0 |   0 ||                     <= Bid. (0=+/-)
---|== === === === === | === === === === === === ===  Ray index Ray mod.
 1 | +/- |   1   0   0 |   0 ||   0  -2   4   2  -6  |       0 |   100 |
 2 | +/- |   0   1   0 |   0 ||   0   2   2   6  10  |       0 |   100 |
```

```
  3 | +/- |   0   0   1 |   0 ||   0  -1  -1  -3  -5    |      0 |    100 |
  4 | +/- |   0   0   0 |   1 ||   1   0   0 -20 -24    |      0 |    100 |

Step. number:          1
Ineq. number:          1
New Tableau:

Tableau R:
 -7 |   0 |   1   1   1 |   1 ||   1   5   7  31  45 <= Norm
 -6 |   0 |  10  10  10 |  10 ||  10  30  46 212 271 <= Modulus (approx.)
 -5 |   0 |   0   0   0 |   0 ||   0   0   0   0   0 <= Redundant
 -4 |   0 |   0   0   0 |   0 ||   1   0   0   0   0 <= Considered
 -3 |   0 |   5   5   5 |   6 ||   6   1   1   0   0 <= NO of (0)
 -2 |   0 |   1   1   1 |   1 ||   1   3   3   3   3 <= NO of (+)
 -1 |   0 |   1   1   1 |   0 ||   0   3   3   4   4 <= NO of (-)
  0 |   1 |   0   0   0 |   0 ||                     <= Bid. (0=+/-)
---|== === === === === | === === === === === === ===
  1 | +/- |   1   0   0 |   0 ||   0  -2   4   2  -6
  2 | +/- |   0   1   0 |   0 ||   0   2   2   6  10
  3 | +/- |   0   0   1 |   0 ||   0  -1  -1  -3  -5
  4 |     |   0   0   0 |   1 ||   1   0   0 -20 -24


Step. number:          2
Ineq. number:          4
New Tableau:

Tableau R:
 -7 |   0 |   1   1   1 |   1 ||   1   5   7  31  45 <= Norm
 -6 |   0 |  10  10  10 |  10 ||  10  30  46 212 271 <= Modulus (approx.)
 -5 |   0 |   0   0   0 |   0 ||   0   0   0   0   0 <= Redundant
 -4 |   0 |   0   0   0 |   0 ||   1   0   0   2   0 <= Considered
 -3 |   0 |   6   6   6 |   9 ||   9   2   2   9   2 <= NO of (0)
 -2 |   0 |   4   4   2 |   3 ||   3   6   8   3   6 <= NO of (+)
 -1 |   0 |   2   2   4 |   0 ||   0   4   2   0   4 <= NO of (-)
  0 |   4 |   0   0   0 |   0 ||                     <= Bid. (0=+/-)
---|== === === === === | === === === === === === ===
```

```
 1 |       |   0   0  -1 |   0 ||   0   1   1   3   5
 2 | +/-   |   3   0   2 |   0 ||   0  -8  10   0 -28
 3 | +/-   |   0   1   2 |   0 ||   0   0   0   0   0
 4 |       |   0   0 -20 |   3 ||   3  20  20   0  28
```

```
Step. number:          3
Ineq. number:          2
New Tableau:

Tableau R:
-7 |   0 |   1   1   1 |   1 ||   1   5   7  31  45 <= Norm
-6 |   0 |  10  10  10 |  10 ||  10  30  46 212 271 <= Modulus (approx.)
-5 |   0 |   0   0   0 |   0 ||   0   0   0   0   0 <= Redundant
-4 |   0 |   0   0   0 |   0 ||   1   3   0   2   0 <= Considered
-3 |   0 |   4   5   0 |   5 ||   5   4   2   5   2 <= NO of (0)
-2 |   0 |   2   1   1 |   2 ||   2   3   4   2   4 <= NO of (+)
-1 |   0 |   1   1   6 |   0 ||   0   0   1   0   1 <= NO of (-)
 0 |   2 |   0   0   0 |   0 ||                     <= Bid. (0=+/-)
---|== === === === === | === === === === === === ===
 1 |     |  -3   0  -2 |   0 ||   0   8 -10   0  28
 2 | +/- |   0   1   2 |   0 ||   0   0   0   0   0
 3 |     |   1   0  -2 |   0 ||   0   0   6   8   4
 4 |     |   5   0 -10 |   2 ||   2   0  30   0 -28
```

```
Step. number:          4
Ineq. number:          3
New Tableau:

Tableau R:
-7 |   0 |   1   1   1 |   1 ||   1   5   7  31  45 <= Norm
-6 |   0 |  10  10  10 |  10 ||  10  30  46 212 271 <= Modulus (approx.)
-5 |   0 |   0   0   0 |   0 ||   0   0   0   0   0 <= Redundant
-4 |   0 |   0   0   0 |   0 ||   1   3   4   2   0 <= Considered
-3 |   0 |   2   4   0 |   4 ||   4   4   4   4   2 <= NO of (0)
-2 |   0 |   2   1   1 |   2 ||   2   2   2   2   3 <= NO of (+)
```

```
-1 |   0 |   2   1   5 |   0 ||   0   0   0   0   1 <= NO of (-)
 0 |   3 |   0   0   0 |   0 ||                     <= Bid. (0=+/-)
---|== === === === === | === === === === === === ===
 1 | +/- |   0   1   2 |   0 ||   0   0   0   0   0
 2 |     |   1   0  -2 |   0 ||   0   0   6   8   4
 3 |     |   5   0 -10 |   2 ||   2   0  30   0 -28
 4 |     |  -1   0  -4 |   0 ||   0   6   0  10  26
 5 |     |  -2   0  -8 |   1 ||   1  12   0   0  28
```

```
Step. number:          5
Ineq. number:          5
New Tableau:
```

```
Tableau R:
-7 |   0 |   1   1   1 |   1 ||   1   5   7  31  45 <= Norm
-6 |   0 |  10  10  10 |  10 ||  10  30  46 212 271 <= Modulus (approx.)
-5 |   0 |   0   0   0 |   0 ||   0   0   0   0   0 <= Redundant
-4 |   0 |   0   0   0 |   0 ||   1   3   4   2   5 <= Considered
-3 |   0 |   2   6   0 |   4 ||   4   4   4   4   5 <= NO of (0)
-2 |   0 |   4   1   1 |   4 ||   4   4   4   4   3 <= NO of (+)
-1 |   0 |   2   1   7 |   0 ||   0   0   0   0   0 <= NO of (-)
 0 |   5 |   0   0   0 |   0 ||                     <= Bid. (0=+/-)
---|== === === === === | === === === === === === ===
 1 | +/- |   0   1   2 |   0 ||   0   0   0   0   0
 2 |     |   1   0  -2 |   0 ||   0   0   6   8   4
 3 |     |  -1   0  -4 |   0 ||   0   6   0  10  26
 4 |     |  -2   0  -8 |   1 ||   1  12   0   0  28
 5 |     |   6   0 -12 |   1 ||   1   0  36  28   0
 6 |     |   1   0  -6 |   1 ||   1   4  10   0   0
```

```
Final solution:

Tableau F:
-7 |   0 |   1   1   1 |   1 ||   1   5   7  31  45 <= Norm
-6 |   0 |  10  10  10 |  10 ||  10  30  46 212 271 <= Modulus (approx.)
```

```
-5 |   0 |   0   0   0 |   0 ||   0   0   0   0   0 <= Redundant
-4 |   0 |   0   0   0 |   0 ||   1   3   4   2   5 <= Considered
-3 |   0 |   2   6   0 |   4 ||   4   4   4   4   5 <= NO of (0)
-2 |   0 |   4   1   1 |   4 ||   4   4   4   4   3 <= NO of (+)
-1 |   0 |   2   1   7 |   0 ||   0   0   0   0   0 <= NO of (-)
 0 |   5 |   0   0   0 |   0 ||                       <= Bid. (0=+/-)
---|== === === === === | === === === === === === ===
 1 | +/- |   0   1   2 |   0 ||   0   0   0   0   0
 2 |     |   1   0  -2 |   0 ||   0   0   6   8   4
 3 |     |  -1   0  -4 |   0 ||   0   6   0  10  26
 4 |     |  -2   0  -8 |   1 ||   1  12   0   0  28
 5 |     |   6   0 -12 |   1 ||   1   0  36  28   0
 6 |     |   1   0  -6 |   1 ||   1   4  10   0   0


(*========================================*)
(*              DUAL EXAMPLE              *)
(*========================================*)

Constraint matrix (bidirectional) Q:
 - First component: 0 <=> bidirectional   <=> +/-
                    1 <=> unidirectional. <=>
Matrix Q:
   (0)   0   1   2   0
   (1)   1   0  -2   0
   (1)  -1   0  -4   0
   (1)  -2   0  -8   1
   (1)   6   0 -12   1
   (1)   1   0  -6   1

Initial tableau:

Tableau I:
-7 |   0 |   1   1   1 |   1 ||   3   3   5  11  19   8 <= Norm
-6 |   0 |  10  10  10 |  10 ||  22  22  41  83 135  62 <= Modulus (approx.)
-5 |   0 |   0   0   0 |   0 ||   0   0   0   0   0   0 <= Redundant
```

```
-4 |   0 |   0   0   0 |   0 ||   0   0   0   0   0   0 <= Considered
-3 |   0 |   6   6   6 |   6 ||   4   4   4   2   2   2 <= NO of (0)
-2 |   0 |   1   1   1 |   1 ||   2   2   2   3   3   3 <= NO of (+)
-1 |   0 |   1   1   1 |   1 ||   2   2   2   3   3   3 <= NO of (-)
 0 |   0 |   0   0   0 |   0 || +/-                     <= Bid. (0=+/-)
---|== === === === === | === === === === === === === ===  Ray index Ray mod.
 1 | +/- |   1   0   0 |   0 ||   0   1  -1  -2   6   1  |     0 |   100 |
 2 | +/- |   0   1   0 |   0 ||   1   0   0   0   0   0  |     0 |   100 |
 3 | +/- |   0   0   1 |   0 ||   2  -2  -4  -8 -12  -6  |     0 |   100 |
 4 | +/- |   0   0   0 |   1 ||   0   0   0   1   1   1  |     0 |   100 |

Step. number:          1
Ineq. number:          1
New Tableau:

Tableau R:
-7 |   0 |   1   1   1 |   1 ||   3   3   5  11  19   8 <= Norm
-6 |   0 |  10  10  10 |  10 ||  22  22  41  83 135  62 <= Modulus (approx.)
-5 |   0 |   0   0   0 |   0 ||   0   0   0   0   0   0 <= Redundant
-4 |   0 |   0   0   0 |   0 ||   1   0   0   0   0   0 <= Considered
-3 |   0 |   4   4   4 |   4 ||   6   2   2   0   0   0 <= NO of (0)
-2 |   0 |   1   1   1 |   1 ||   0   2   2   3   3   3 <= NO of (+)
-1 |   0 |   1   1   1 |   1 ||   0   2   2   3   3   3 <= NO of (-)
 0 |   1 |   0   0   0 |   0 || +/-                     <= Bid. (0=+/-)
---|== === === === === | === === === === === === === ===
 1 | +/- |   1   0   0 |   0 ||   0   1  -1  -2   6   1
 2 | +/- |   0   0   0 |   1 ||   0   0   0   1   1   1
 3 | +/- |   0  -2   1 |   0 ||   0  -2  -4  -8 -12  -6


Step. number:          2
Ineq. number:          2
New Tableau:

Tableau R:
-7 |   0 |   1   1   1 |   1 ||   3   3   5  11  19   8 <= Norm
-6 |   0 |  10  10  10 |  10 ||  22  22  41  83 135  62 <= Modulus (approx.)
```

```
-5 |   0 |   0   0   0 |   0 ||   0   0   0   0   0   0 <= Redundant
-4 |   0 |   0   0   0 |   0 ||   1   2   0   0   0   0 <= Considered
-3 |   0 |   3   3   3 |   4 ||   6   4   2   0   2   0 <= NO of (0)
-2 |   0 |   2   2   1 |   1 ||   0   2   2   3   3   4 <= NO of (+)
-1 |   0 |   1   1   2 |   1 ||   0   0   2   3   1   2 <= NO of (-)
 0 |   2 |   0   0   0 |   0 || +/-                     <= Bid. (0=+/-)
---|== === === === === | === === === === === === === ===
 1 | +/- |   0   0   0 |   1 ||   0   0   0   1   1   1
 2 |     |   0   2  -1 |   0 ||   0   2   4   8  12   6
 3 | +/- |   2  -2   1 |   0 ||   0   0  -6 -12   0  -4


Step. number:         3
Ineq. number:         3
New Tableau:

Tableau R:
-7 |   0 |   1   1   1 |   1 ||   3   3   5  11  19   8 <= Norm
-6 |   0 |  10  10  10 |  10 ||  22  22  41  83 135  62 <= Modulus (approx.)
-5 |   0 |   0   0   0 |   0 ||   0   0   0   0   0   0 <= Redundant
-4 |   0 |   0   0   0 |   0 ||   1   2   3   0   0   0 <= Considered
-3 |   0 |   3   2   2 |   3 ||   5   3   3   1   1   0 <= NO of (0)
-2 |   0 |   1   3   0 |   1 ||   0   2   2   3   3   4 <= NO of (+)
-1 |   0 |   1   0   3 |   1 ||   0   0   0   1   1   1 <= NO of (-)
 0 |   3 |   0   0   0 |   0 || +/-                     <= Bid. (0=+/-)
---|== === === === === | === === === === === === === ===
 1 | +/- |   0   0   0 |   1 ||   0   0   0   1   1   1
 2 |     |  -2   2  -1 |   0 ||   0   0   6  12   0   4
 3 |     |   4   2  -1 |   0 ||   0   6   0   0  36  10


Step. number:         4
Ineq. number:         6
New Tableau:

Tableau R:
-7 |   0 |   1   1   1 |   1 ||   3   3   5  11  19   8 <= Norm
```

```
 -6 |   0 |  10  10  10 |  10 ||  22  22  41  83 135  62 <= Modulus (approx.)
 -5 |   0 |   0   0   0 |   0 ||   0   0   0   0   0   0 <= Redundant
 -4 |   0 |   0   0   0 |   0 ||   1   2   3   0   0   4 <= Considered
 -3 |   0 |   1   1   1 |   2 ||   5   3   3   1   1   2 <= NO of (0)
 -2 |   0 |   2   4   0 |   1 ||   0   2   2   3   3   3 <= NO of (+)
 -1 |   0 |   2   0   4 |   2 ||   0   0   0   1   1   0 <= NO of (-)
  0 |   6 |   0   0   0 |   0 || +/-                     <= Bid. (0=+/-)
 ---|== === === === === | === === === === === === === ===
  1 |     |   0   0   0 |   1 ||   0   0   0   1   1   1
  2 |     |  -2   2  -1 |  -4 ||   0   0   6   8  -4   0
  3 |     |   4   2  -1 | -10 ||   0   6   0 -10  26   0
```

```
Step. number:          5
Ineq. number:          4
New Tableau:
```

```
Tableau R:
 -7 |   0 |   1   1   1 |   1 ||   3   3   5  11  19   8 <= Norm
 -6 |   0 |  10  10  10 |  10 ||  22  22  41  83 135  62 <= Modulus (approx.)
 -5 |   0 |   0   0   0 |   0 ||   0   0   0   0   0   0 <= Redundant
 -4 |   0 |   0   0   0 |   0 ||   1   2   3   5   0   4 <= Considered
 -3 |   0 |   1   1   1 |   1 ||   4   2   2   2   0   2 <= NO of (0)
 -2 |   0 |   2   3   0 |   1 ||   0   2   2   2   3   2 <= NO of (+)
 -1 |   0 |   1   0   3 |   2 ||   0   0   0   0   1   0 <= NO of (-)
  0 |   4 |   0   0   0 |   0 || +/-                     <= Bid. (0=+/-)
 ---|== === === === === | === === === === === === === ===
  1 |     |   0   0   0 |   1 ||   0   0   0   1   1   1
  2 |     |  -2   2  -1 |  -4 ||   0   0   6   8  -4   0
  3 |     |   4   2  -1 |   0 ||   0   6   0   0  36  10
  4 |     |   2   6  -3 | -20 ||   0   8  10   0  28   0
```

```
Step. number:          6
Ineq. number:          5
New Tableau:
```

```
Tableau R:
-7 |   0 |   1   1   1 |   1 ||   3   3   5  11  19   8 <= Norm
-6 |   0 |  10  10  10 |  10 ||  22  22  41  83 135  62 <= Modulus (approx.)
-5 |   0 |   0   0   0 |   0 ||   0   0   0   0   0   0 <= Redundant
-4 |   0 |   0   0   0 |   0 ||   1   2   3   5   6   4 <= Considered
-3 |   0 |   1   1   1 |   2 ||   6   2   2   2   3   2 <= NO of (0)
-2 |   0 |   2   5   0 |   1 ||   0   4   4   4   3   4 <= NO of (+)
-1 |   0 |   3   0   5 |   3 ||   0   0   0   0   0   0 <= NO of (-)
 0 |   5 |   0   0   0 |   0 || +/-                     <= Bid. (0=+/-)
---|== === === === === | === === === === === === === ===
 1 |     |   0   0   0 |   1 ||   0   0   0   1   1   1
 2 |     |   4   2  -1 |   0 ||   0   6   0   0  36  10
 3 |     |   2   6  -3 | -20 ||   0   8  10   0  28   0
 4 |     |  -2   2  -1 |   0 ||   0   0   6  12   0   4
 5 |     |  -6  10  -5 | -24 ||   0   4  26  28   0   0


Final solution:

Tableau F:
-7 |   0 |   1   1   1 |   1 ||   3   3   5  11  19   8 <= Norm
-6 |   0 |  10  10  10 |  10 ||  22  22  41  83 135  62 <= Modulus (approx.)
-5 |   0 |   0   0   0 |   0 ||   0   0   0   0   0   0 <= Redundant
-4 |   0 |   0   0   0 |   0 ||   1   2   3   5   6   4 <= Considered
-3 |   0 |   1   1   1 |   2 ||   6   2   2   2   3   2 <= NO of (0)
-2 |   0 |   2   5   0 |   1 ||   0   4   4   4   3   4 <= NO of (+)
-1 |   0 |   3   0   5 |   3 ||   0   0   0   0   0   0 <= NO of (-)
 0 |   5 |   0   0   0 |   0 || +/-                     <= Bid. (0=+/-)
---|== === === === === | === === === === === === === ===
 1 |     |   0   0   0 |   1 ||   0   0   0   1   1   1
 2 |     |   4   2  -1 |   0 ||   0   6   0   0  36  10
 3 |     |   2   6  -3 | -20 ||   0   8  10   0  28   0
 4 |     |  -2   2  -1 |   0 ||   0   0   6  12   0   4
 5 |     |  -6  10  -5 | -24 ||   0   4  26  28   0   0
```

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